An Introduction to Formal (Infinity-)Category Theory via Modules

1. Introduction

Abstract

At the heart of formal category theory is the construction of higher categories. However, structures such as 2-categories, or more generally bicategories (and their higher dimensional analogues), are often insufficient for encoding a number of important abstract constructions prevalent in category theory. Although notions such as adjunctions and equivalences can be defined purely algebraically internal to any bicategory, the naive definition of fully-faithfulness and of Kan extensions often fails to encapsulate the right properties in enriched settings. This failure stems from the inability to discuss representability, or equivalently yoneda constructions, internal to a general bicategory.

In this talk we will provide a brief introduction to the framework of formal category theory via pro-arrow equipments which fixes this issue. After this brief introduction we will explore how this work generalizes to the construction of a formal theory of ∞-categories due to Riehl and Verity (with further recent work by Ruit) which provides a suitable definition of Kan extensions internal to an arbitrary ∞-cosmos (and hence any well-behaved model of ∞-categories).

This note accompanies a brief talk on the formal theory of ∞-category theory at the UIUC Graduate Homotopy Theory Seminar, motivated primarily by the work of Riehl and Verity on ∞-cosmoi in 24 and 5, as well as the classical theory of formal category theory via pro-arrow equipments discussed by Verity in6 with additional extensions by Shulman in3. Although this note does not cover the theory of pro-arrow equipments internal to ∞-categories, as constructed by Ruit in1, Ruit's work is a large motivator for the speaker's interest in this area.

As discussed at length in a number of the above sources, one of the main goals of formal category theory is to be able to abstract and encapsulate the essential aspects of category theory in its many guises, such as enriched category theory and fibrant category theory. A first attempt at this can be seen in the structure of 2-categories which are rich enough to formalize notions of adjunctions and equivalences. However, it was found that such structures alone were insufficient for the development of a great number of important categorical concepts and tools. For instance, one can formulate a definition of Kan extensions internal to a 2-category, but this would only produce the naive definition of Kan extensions rather than the more prolific and useful notion of pointwise Kan extensions.

Similarly, although one can attempt to representably define fully-faithfulness of functors, often this will produce the wrong notion, for example when considering enriched concepts.

But why should we care about pointwise Kan extensions over the naive notion that can be defined internal to any 2-category? Some motivation from homotopy theory is that pointwise Kan extensions are needed to obtain functorial derived functors, while the result that Quillen adjunctions descend to adjunctions on the level of homotopy categories also relies on this pointwise (or more strictly absolute) notion of Kan extensions.

It turns out that the thing missing from these structures is exactly a theory which can internally discuss the notion of representability and yoneda embeddings. Generalizing, one can ask that we can encapsulate a suitable notion of *profunctors* $A \nrightarrow B$, which are classically represented by bifunctors $B^{op} \times A \to \mathsf{Set}$ in ordinary category theory, or $B^{op} \times A \to \mathcal{V}$ in a $\mathcal{V}$-enriched context. The canonical example of such a functor (resp. $\mathcal{V}$-functor) is given by the hom-bifunctor

$$\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathsf{Set}$$

for a category $\mathcal{C}$, which is adjoint to the yoneda embedding $\yo : \mathcal{C} \to \mathsf{Ps}(\mathcal{C})$. In this note we aim to provide a brief intuition for structures that behave like profunctors more generally, and how we can extend them to ∞-categorical contexts.

2. Classical Theory of Pro-arrow Equipments

Pro-arrow equipments are an additional structure on a bicategory introduced by Wood, and studied further by Verity in6, in order to encapsulate the structure of profunctors present in the canonical bicategory Cat_2.

Definition 1 (Equipments on Bicategories).

An **equipment** on a bicategory $\mathcal{K}$ consists of another bicategory $\mathcal{B}$ along with an identity on objects bifunctor

$$(-)_* : \mathcal{K}^{co} \to \mathcal{B}$$

such that for any 1-cell $f : a \to b$ in $\mathcal{K}$, f_* admits a right adjoint f^* in $\mathcal{B}$. An equipment is called a **pro-arrow equipment** if in addition $(-)_*$ is locally fully faithful.

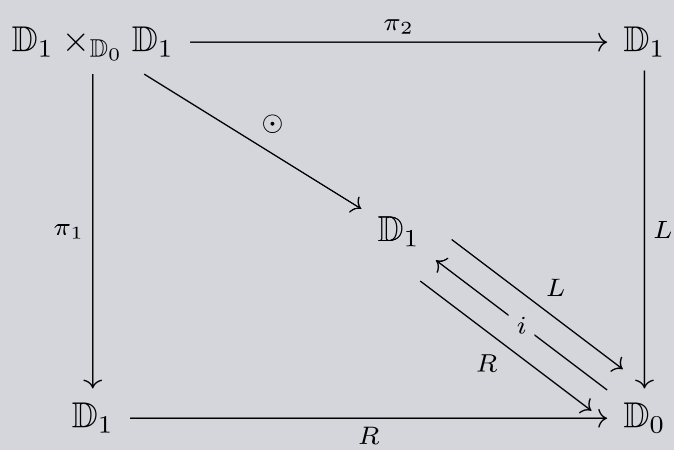
However, we will not explore these definitions, as given here, much further in this note, instead preferring Verity's reformulation in terms of the theory of double categories which we now briefly review. For this perspective we will primarily follow Shulman's exposition in3, where he refers to these structures as *framed bicategories*. These structures have important applications outside of just the formal theory of category theory, such as in the study of *parametrized spectra* and in our ∞-categorical setting to come, *parametrized ∞-categories* which provide a way of studying enriched ∞-categories.

2.1. Double Categories and Fibrations

As in3 and4 we will be restricting to what in double-categorical literature is often referred to as *pseudo double categories* when speaking of *double categories* in this note. Intuitively, one can think of a double category as a pair of a vertical category and a horizontal category with squares relating their morphisms such that the composition in the horizontal category need not be strict (and as we will see later with *virtual double categories*, may not even be defined).

Definition 2 (Double Category).

A **double category** $\mathbb{D}$ is an internal category object in Cat_2:



equipped with natural isomorphisms

$$a : (M \odot N) \odot P \xrightarrow{\cong} M \odot (N \odot P)$$

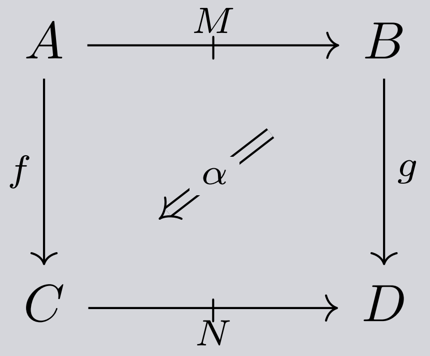
$$l : (i_A \odot M) \xrightarrow{\cong} M \quad \text{and} \quad r : (M \odot i_B) \xrightarrow{\cong} M$$

satisfying appropriate coherence diagrams, and which are identities under whiskering with L or R.

Here we write $\odot$ in diagrammatic ordering, suggesting our emphasis on the perspective that pro-arrows in a double category represent modules.

Example 3 (Double Category of Squares in a 2-category).

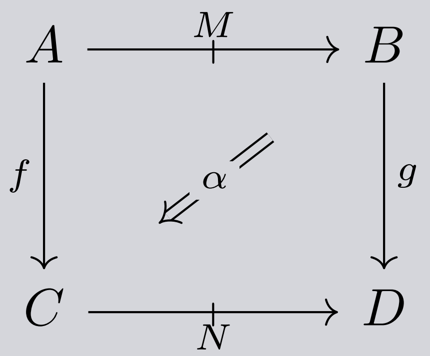
Let $\mathcal{B}$ be a 2-category. Then we have a natural double category $\mathsf{Sq}(\mathcal{B})$ whose category $\mathcal{B}_0$ is the underlying category of $\mathcal{B}$, and whose category $\mathcal{B}_1$ has as objects 1-cells in $\mathcal{B}$ and as morphisms squares



Horizontal and vertical composition is given by pasting 2-cells.

Example 4 (Double Category of Modules).

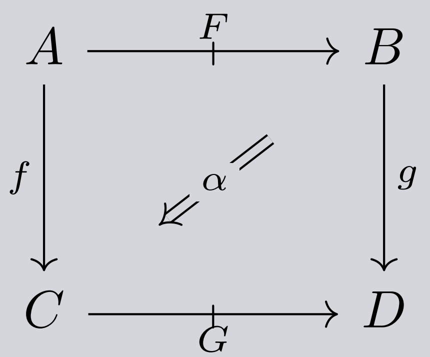
Let Mod be the double category with $\mathsf{Mod}_0 = \mathsf{Ring}$, the category of (unital) rings, and Mod_1 the category with objects bimodules and morphisms bimodule homomorphisms. Explicitly, if A, B, C, and D are rings, and $f : A \to C$ and $g : B \to D$ are ring homomorphisms, a square



represents a map of A-B bimodules $M \to {}_fN_g$, where ${}_fN_g$ is the double restriction of scalars for N along both ring maps. Horizontal composition is given by tensor product of modules.

Example 5 (Profunctors).

The double category Prof has $\mathsf{Prof}_0 = \mathsf{Cat}$, the category of small categories, and Prof_1 the category with objects profunctors $F : \mathcal{A}^{op} \times \mathcal{B} \to \mathsf{Set}$, with a 2-cell



given by a natural transformation $F \Rightarrow (G \circ (g, f^{op}))$. Horizontal composition on profunctors is given by the co-end formula

$$F \odot G(a, c) = \int^{b:\mathcal{B}} F(a, b) \otimes G(b, c)$$

where $\otimes$ on the right is standard tensor on Set given by cartesian product, or more general for $\mathcal{V} - \mathsf{Prof}$, is the chosen tensor in $\mathcal{V}$.

An important piece in the theory of pro-arrow equipments, especially when we inevitably move on to the theory of equipments as a descriptor for modules between ∞-categories, is the notion of fibered categories. Briefly, a *Grothendieck fibration* $p : \mathcal{E} \to \mathcal{B}$ between categories is a functor such that for every $e \in \mathcal{E}$ and every morphism $f : b \to p(e)$ in $\mathcal{B}$, there exists a lift $\varphi : e' \to e$ making the diagram

$$\begin{array}{ccc} \mathcal{E}_{/\varphi} & \xrightarrow{\text{cod}} & \mathcal{E}_{/e} \\ \downarrow{\scriptstyle p} & \lrcorner & \downarrow{\scriptstyle p} \\ \mathcal{B}_{/p(\varphi)} & \xrightarrow[\text{cod}]{} & \mathcal{B}_{/p(e)} \end{array}$$

a pullback in Cat. Intuitively this says that φ universally lifts factorizations through f for maps comin from $\mathcal{E}$. Dually, an *opfibration* $p : \mathcal{E} \to \mathcal{B}$ is a functor such that for every $e \in \mathcal{E}$ and $f : p(e) \to b$ in $\mathcal{B}$, there exists a lift $\varphi : e \to e'$ making the diagram

$$\begin{array}{ccc} \mathcal{E}_{\varphi/} & \xrightarrow{\text{dom}} & \mathcal{E}_{e/} \\ \downarrow{\scriptstyle p} & \lrcorner & \downarrow{\scriptstyle p} \\ \mathcal{B}_{p(\varphi)/} & \xrightarrow[\text{dom}]{} & \mathcal{B}_{p(e)/} \end{array}$$

a pullback in Cat.

Two important results for fibrations which are relevant to this note are their correspondence to pseudo-functors under the Grothendieck construction and the relation between fibrations and op-fibrations in terms of adjunctions in the total category.

Theorem 6 (Grothendieck Construction [Shu,3.8]).

Given a category $\mathcal{B}$, we have an equivalence

$$\mathsf{Fib}_{\mathcal{B}} \simeq [\mathcal{B}^{op}, \mathsf{Cat}]$$

where the category on the left has as objects Grothendieck fibrations and as morphisms maps in the slice $\mathsf{Cat}/\mathcal{B}$ which preserve cartesian morphisms, and the category on the right has as objects pseudo-functors and as morphisms strong transformations, where $\mathcal{B}$ is viewed as a discrete bicategory.

Proposition 7 (Adjoint Relation Between Fibrations and Op-Fibrations [Shu,3.9]).

A fibration $p : \mathcal{E} \to \mathcal{B}$ is an opfibration if and only if for every $f : A \to B$ in $\mathcal{B}$, the natural functor $f^* : p^{-1}(B) \to p^{-1}(A)$ has a left adjoint $f_! : p^{-1}(A) \to p^{-1}(B)$.

Proof.
Let $f : A \to B$ be a map in $\mathcal{B}$. Note that by the classification of cartesian lifts applied to identity fillings, we have a natural bijection between maps $Y \to X$ in $\mathcal{E}$ lying over f and maps $Y \to f^*X$ in $p^{-1}(A)$. On the other hand, the dual property for opcartesian lifts gives a bijection between maps $Y \to X$ over f and maps $f_!Y \to X$ in $p^{-1}(B)$. In other words, we have the natural bijection

$$p^{-1}(B)(f_!Y, X) \cong p^{-1}(A)(Y, f^*X)$$

so the existence of opcartesian lifts is equivalent to the existence of right adjoints, and vice-versa if we start with an opfibration.

□

2.2. Pro-arrow Equipments

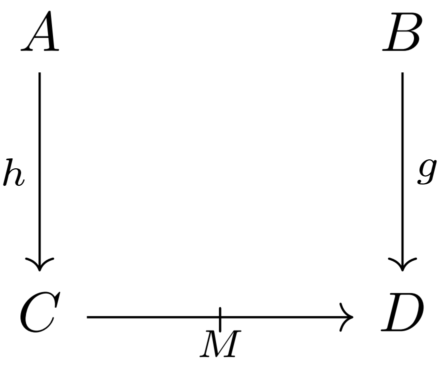
In addition to the existence of pro-arrows, which are our horizontals arrows in any double category, we want suitable assumptions on our double categories so that they behave like profunctors in order to encapsulate an appropriate theory of kan extensions. This behaviour is encapsulated in the language of fibrations reviewed above.

Definition 8 (Pro-arrow Equipment).

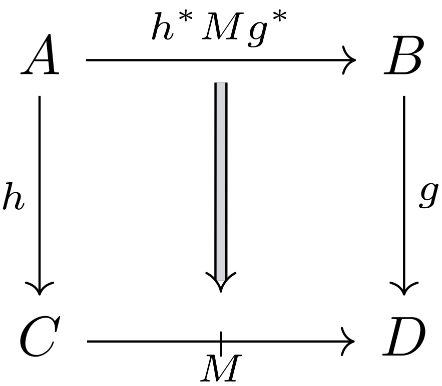
A double category $\mathbb{D} = (\mathbb{D}_1, \mathbb{D}_0, \odot, L, R, i)$ is said to be a **pro-arrow equipment** on $\mathbb{D}_0$ $(L, R) : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is a bifibration.

This structure equivalently encapsulates the idea of base change, which is terminology of bimodules instead of pro-arrows when talking about double categories, and in particular pro-arrow equipments.

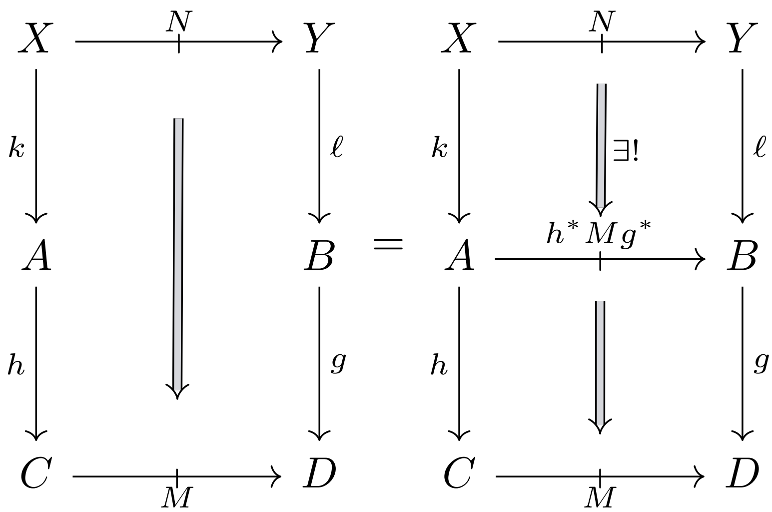
Let's unpack what it means for $(L, R) : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ to be a bifibration. Since such a functor being a bifibration is equivalent to it being a fibration or a opfibration (c.f. [Shu,4.1]) we'll just consider the case of fibration. Explicitly we're saying that any cup



can be universally filled to a square



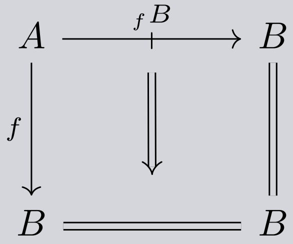
in the sense that any rectangle uniquely factors through it:



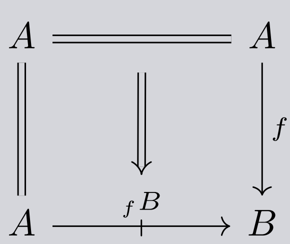
This structure is equivalent to the existence of companions and conjoints for all vertical morphisms in the double category $\mathbb{D}$ (c.f. [Shu, 4.1]).

Definition 9 (Companions and Conjoints).

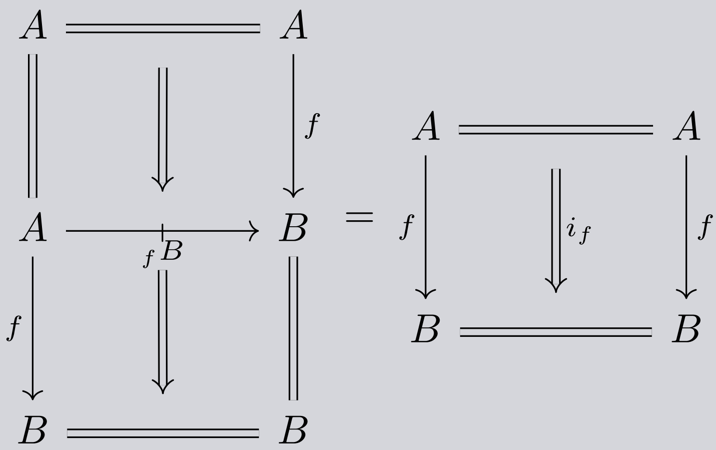
Let $\mathbb{D}$ be a double category and $f : A \to B$ a vertical arrow. A **companion** for f is a horizontal arrow ${}_fB : A \nrightarrow B$ together with squares



and



satisfying



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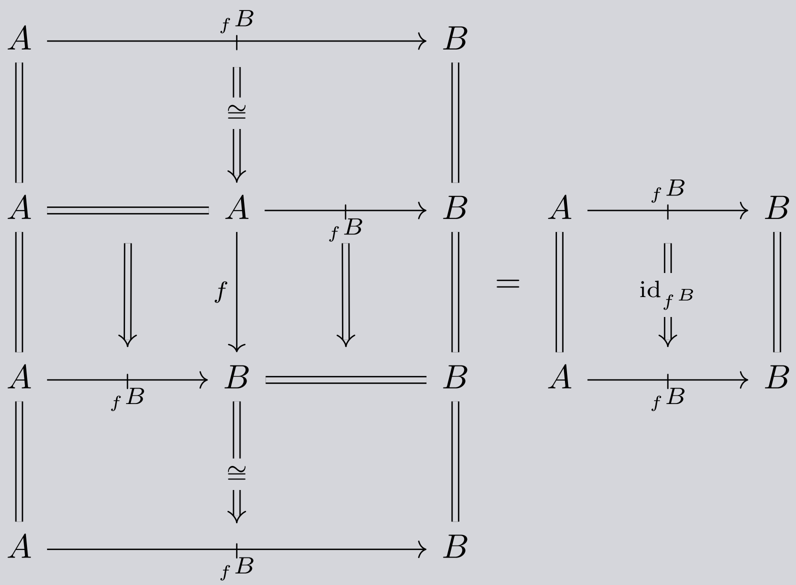


A **conjoint** of f is a horizontal morphism $B_f : B \nrightarrow A$ together with dual squares and relations.

It is easy to see that companions and conjoints arise from the characterizing property of (L, R) being a fibration and opfibration, respectively. Explicitly, for $f : A \to B$ is a vertical morphism, then ${}_f B = f^* i_B \mathrm{id}^*_B$, where $i_B = i(B)$ is the identity at B in $\mathbb{D}_1$ under horizontal composition. Similarly, $B_f = \mathrm{id}^*_A i_B f^*$. Going the other way, a filling $h^* M g^*$ as above the definition can be precisely obtained from companions and conjoints as the composite, or tensor, ${}_hC \odot M \odot D_g$, while the dual filling for opfibrations, $h_! M g_!$, is given by $C_h \odot M \odot {}_gD$.

These are the structures that will allow us to formalize and generalize weighted colimits, fully-faithfulness, and (pointwise) Kan extensions internally to a pro-arrow equipment.

Example 10 (Double Category of Modules).

Mod is a pro-arrow equipment, where for ring homomorphisms $f : A \to B$ and $g : C \to D$ as well as a (B, D) bimodule M, f^*Mg^* is exactly the (A, C) bimodule obtained by restriction. Equivalently, $f^*Mg^* = f^*C \otimes_C M \otimes_D Dg^*$. The previous formulas also tells us that this forces for N a (A, C) bimodule that

$$f_! N g_! = Bf^* \otimes_A N \otimes_C g^* D$$

which is the usual base-change formula for bimodules.

Example 11 (Double Category of Profunctors).

Prof is a pro-arrow equipment where for functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{C} \to \mathcal{D}$, and a profunctor $H : \mathcal{D}^{op} \times \mathcal{B} \to \mathsf{Set}$, its pullback F^*HG^* is given by the composite

$$H \circ G^{op} \times F : \mathcal{C}^{op} \times \mathcal{A} \to \mathsf{Set}$$

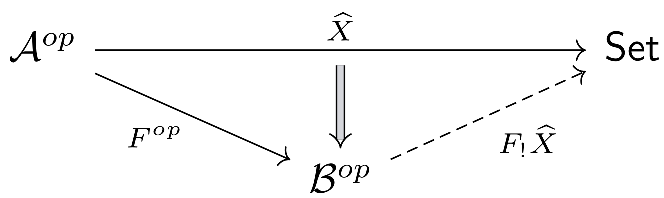
On the other hand, if $K : \mathcal{C}^{op} \times \mathcal{A} \to \mathsf{Set}$ is another profunctor, then $F_! K G_!$ is given by pointwise left Kan extension:

$$F_! K G_!(d, b) = (\mathcal{B}_F \odot K \odot {}_G\mathcal{D})(d, b) = \int^{(c,a):\mathcal{C}^{op} \times \mathcal{A}} \mathcal{B}(F(a), b) \times K(c, a) \times \mathcal{D}(d, G(c))$$

Observe that if $F : \mathcal{A} \to \mathcal{B}$ is any functor, then the profunctor ${}_F\mathcal{B} : \mathcal{B}^{op} \times \mathcal{A} \to \mathsf{Set}$ under adjunction corresponds to a functor $\mathcal{A} \to \mathsf{Ps}(\mathcal{B})$ given by sending $a \in \mathcal{A}$ to $よ^{F(a)} = \mathcal{B}(-, F(a))$. By the universal property of $\mathsf{Ps}(\mathcal{A})$ this corresponds to an accessible functor

$$F_! : \mathsf{Ps}(\mathcal{A}) \to \mathsf{Ps}(\mathcal{B}), \quad \widehat{X} \mapsto F_! \widehat{X}(b) = \int^{a:\mathcal{A}} \widehat{X}(a) \times \mathcal{B}(b, F(a))$$

given by pointwise left Kan extension along F,



On the other hand, the profunctor $\mathcal{B}_F : \mathcal{A}^{op} \times \mathcal{B} \to$ Set under adjunction gives $\mathcal{B} \to \mathsf{Ps}(\mathcal{A})$ where $b \in \mathcal{B}$ is sent to $\mathcal{B}(F(-), b)$. By the universal property of $\mathsf{Ps}(\mathcal{B})$ this corresponds to an accessible functor

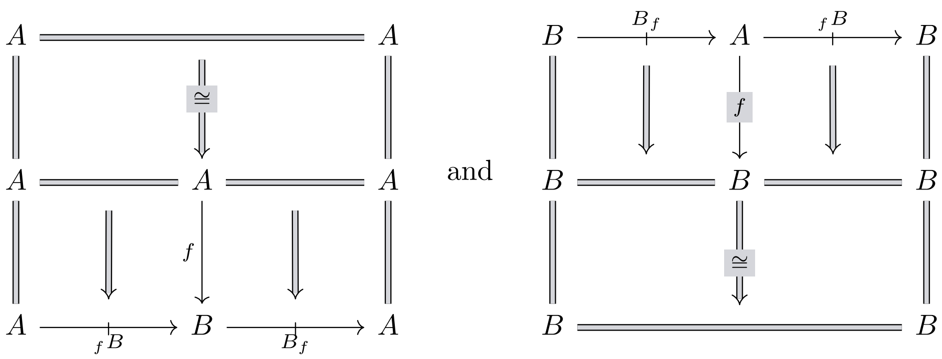
$$F^* : \mathsf{Ps}(\mathcal{B}) \to \mathsf{Ps}(\mathcal{A}), \quad \widehat{X} \mapsto \widehat{X} \circ F^{op}$$

and as the reader might observe, this gives us an adjoint pair $F_! \dashv F^*$. This observation is actually a general phenomenon of pro-arrow equipments.

Proposition 12 (Adjoint Pair of Pro-arrows [Shu, 5.3]).

If $f : A \to B$ is a vertical arrow in a pro-arrow equipment $\mathbb{D}$, then ${}_fB \dashv B_f$ internal to the horizontal bicategory $\mathsf{Hor}(\mathbb{D})$.

Proof.
We can explicit construct the unit and co-unit of this adjunction as the composites



with the triangle identities following from the defining identities for conjoints and companions.

□

This can be viewed as a special case of the following more general result.

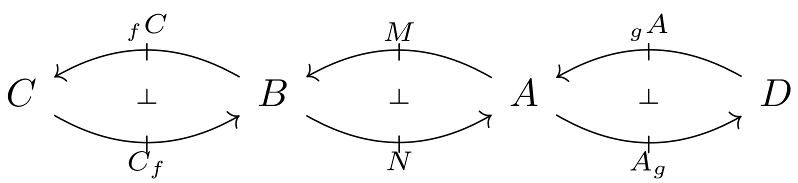
Proposition 13 (Preservation of Adjoint Pairs under Base Change).

Let $M \dashv N$ be an adjoint pair in $\mathsf{Hor}(\mathbb{D})$, with $M : A \nrightarrow B$ and $N : B \nrightarrow A$. If $f : B \to C$ and $g : D \to A$ are vertical morphisms, then $g^* M f_! \dashv f_! N g^*$.

Proof.
Observe that

$$g^* M f_! = {}_gA \odot M \odot {}_fC \quad \text{and} \quad f_! N g^* = C_f \odot N \odot A_g$$

so we have the composite

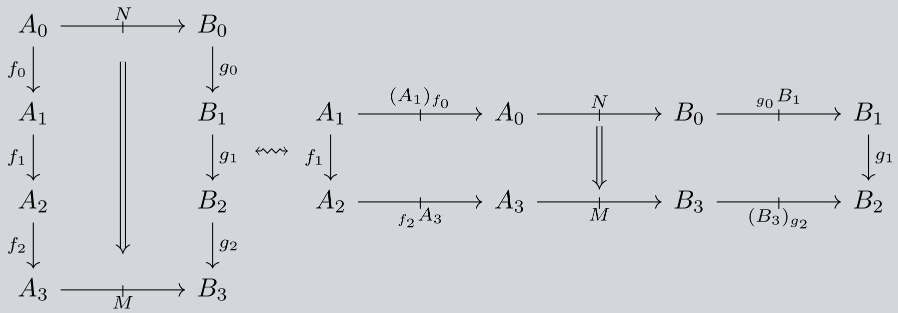


where the result now follows from the fact that adjoints compose in an arbitrary bicategory. □

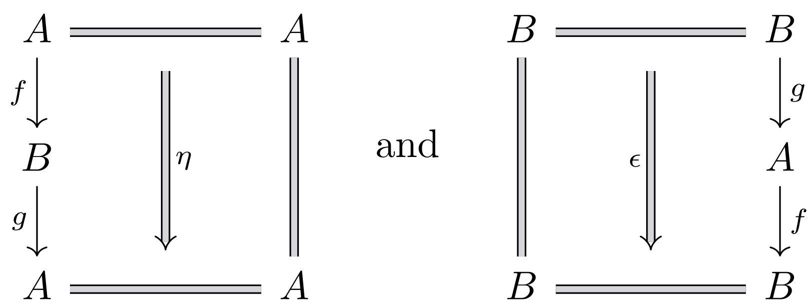
However, if we go with the original philosophy of pro-arrow equipments we should be thinking of the horizontal arrows as a tool for doing category theory in the 2-category Ver($\mathbb{D}$). A key result in this direction is the following lemma describing how we can translate between horizontal and vertical composites using companions and conjoints.

Lemma 14 (Central Lemma).

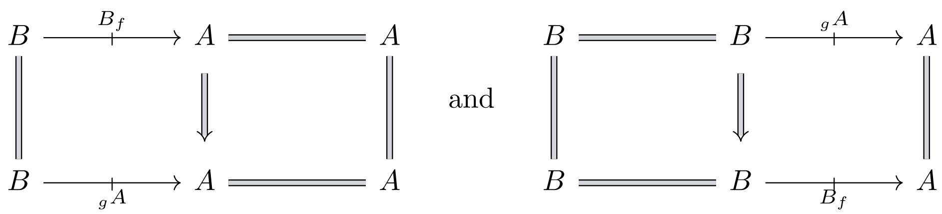
Given a pro-arrow equipment $\mathbb{D}$, there is a natural bijection between squares of the following forms



This equivalence is given by pasting with the squares defining companions and conjoints. This immediately allows us to realize an adjunction in the vertical 2-category Ver($\mathbb{D}$), which takes the form



can equivalently be phrased as squares

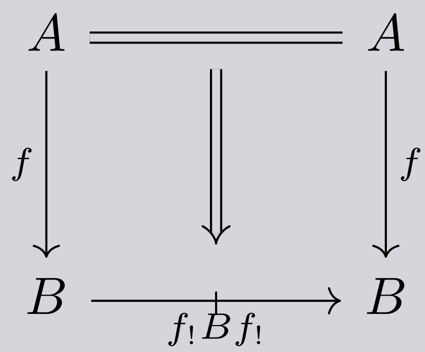


Using our analogy with profunctors we can think of B_f as representing the hom $B(f-,-)$, while ${}_gA$ as $A(-,g-)$. Tracing through how the triangle identities transform shows that we exactly obtain squares which are vertical isomorphisms, given a formal correspondence between internal and hom characterizations of adjunctions.

With this analogy in mind we can give the following definition of fully-faithfulness:

Definition 15 (Fully-Faithful Vertical Morphism in Pro-arrow Equipment).

A vertical morphism $f : A \to B$ in a pro-arrow equipment $\mathbb{D}$ is said to be fully-faithful if and only if the canonical square



is an isomorphism in $\mathbb{D}_1$.

Finally, before moving into our ∞-categorical story lets discuss the main advertised benefit of the use of pro-arrow equipments in formal category theory: the ability to describe the *right* kind of Kan extensions. We will do this through the language of weighted (co)limits. In the context of pro-arrow equipments the pro-arrows serve as our weights, while the vertical arrows are what we're taking (co)limits of.

In order to define this for convenience we could require more structure on our double category $\mathbb{D}$, which classically corresponds to the requirement that we have enough (right) extensions and lifts. Explicitly, we may require that the bicategory $\mathsf{Hor}(\mathbb{D})$ is closed, which is to say its composition sits in a triple of adjunctions

$$\mathsf{Hor}(\mathbb{D})(M \odot N, P) \simeq \mathsf{Hor}(\mathbb{D})(M, N \triangleright P) \simeq \mathsf{Hor}(\mathbb{D})(N, P \triangleleft M)$$

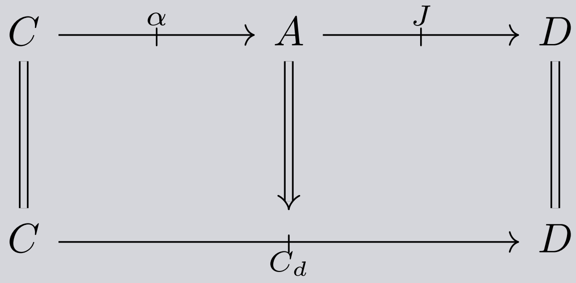
The first equivalence is the statement that we have enough right lifts while the equivalence with the second hom is the statement that we have enough right extensions. General properties of adjoints then imply the for $f : A \to B$, $g : C \to D$, and $M : B \nrightarrow D$,

$$_fM_g \cong (_gD \triangleleft M) \triangleright B_f$$

However, we can also define it in terms of the universal property for local right extensions and lifts.

Definition 16 (Weighted (co)limits in a Pro-arrow Equipment).

Let $d : D \to C$ and $J : A \nrightarrow D$ be a vertical and horizontal arrow in a pro-arrow equipment $\mathbb{D}$. A vertical arrow $\ell : A \to C$ is a **J-weighted colimit of d** if its conjoint $C_\ell : C \nrightarrow A$ represents J-cones under d, or in other words is a right lifting of the conjoint C_d along J. Explicitly, this means that we have a square $C_\ell \odot J \Rightarrow C_d$ such that any square



factors through $C_\ell \odot J \Rightarrow C_d$ via a square $C_\ell \Rightarrow \alpha$.

In $\mathsf{Prof}(\mathcal{V})$ taking $A = \mathbb{1}$ with hom object the monoidal unit, we obtain the usual definition of a weighted colimit in a $\mathcal{V}$-category C for a $\mathcal{V}$-diagram $d : D \to C$. Indeed, in this case a weight $J : A \twoheadrightarrow D$ is the same as a $\mathcal{V}$-functor $J : D^{op} \to \mathcal{V}$, while the universal property says that we have a natural isomorphism

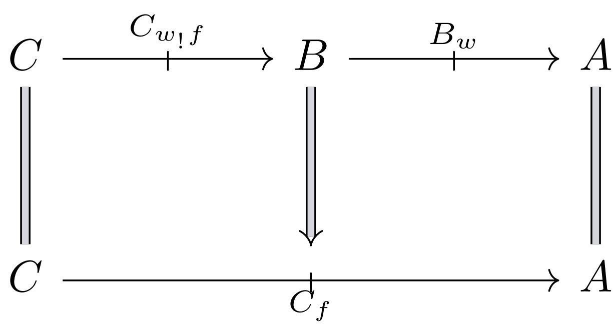
$$C(\ell, -) \cong \int_{d':D} C(d, d')^{Jd'} = [D^{op}, \mathsf{Set}](J, C(d(-), -))$$

We can finally give the right definition of (pointwise) left and right Kan extensions in this context.

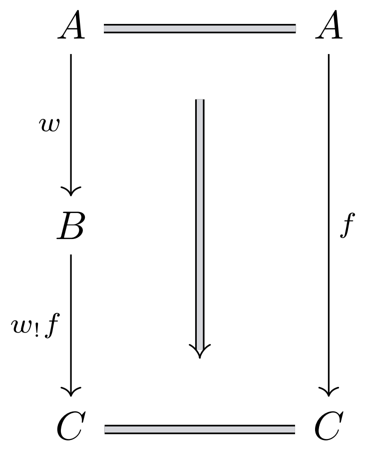
Definition 17 (Pointwise Kan Extension).

If $f : A \to C$ and $w : A \to B$ are vertical arrows in $\mathbb{D}$, then the **pointwise left Kan extension** of f along w, if it exists, is the colimit of f weighted by the conjoint $w_! B = B_w : B \twoheadrightarrow A$.

Let's unpack this definition in the context of our above work. Explicitly, this says that a pointwise left Kan extension of $f : A \to C$ along $w : A \to B$ should be a vertical arrow $w_! f : B \to C$ together with a universal square



Using Lemma 14 (Central Lemma) we can rephrase this as a square



which is the usual description of a left Kan extension, except now the weighted colimit requirement has encoded the correct universal property to ensure that this Kan extension is pointwise.

Explicitly, if $\mathbb{D} = \mathsf{Prof}(\mathcal{V})$ with $\mathcal{V}$ a complete and cocomplete monoidal closed category, then for $\mathcal{V}$-functors $F : \mathcal{A} \to \mathcal{C}$ and $w : \mathcal{A} \to \mathcal{B}$, a left Kan extension of F along w (if it exists) is a $\mathcal{V}$-functor $w_! F : \mathcal{B} \to \mathcal{C}$ together with a universal $\mathcal{V}$-natural transformation

$$\Lambda : \mathcal{C}(w_! F(-), -) \odot \mathcal{B}(w(-), -) \Rightarrow \mathcal{C}(F(-), -)$$

or after adjunction, since $\mathcal{V}$ is (co)complete and closed, a $\mathcal{V}$-natural isomorphism

$$\Lambda : \mathcal{C}(w_! F(-), -) \Rightarrow \mathcal{B}(w(-), -) \triangleright \mathcal{C}(F(-), -) = \int_{a:\mathcal{A}} \mathcal{V}(\mathcal{B}(w(a), -), \mathcal{C}(F(a), -))$$

which is the standard definition of a $\mathcal{V}$-enriched pointwise left Kan extension.

Main takeaway: In order to define a formal notion of weighted colimits, fully-faithfulness, or pointwise Kan extensions, one needs a formal theory encoding representability internal to your structure.

3. Modules Between ∞-Categories

Now that we're more familiar with the classical theory of formal category theory in terms of pro-arrow equipments we can move on to the formal category theory of ∞-categories. In this section we will begin by introducing the notion ∞-cosmos due to Riehl and Verity[7] before introducing the necessary language of (co)cartesian fibrations[5] in order to introduce modules between ∞-categories[4]. Finally, we will conclude with a brief discussion of how these modules between ∞-categories let us encode Kan extensions in ∞-cosmoi.

3.1. A Brief Introduction to ∞-Cosmoi

This note will contain only details of relevance and interest to the current topic on ∞-cosmoi. For a more complete description we refer to Riehl and Verity's text[7]. We will also not delve into the details of the quasi-categorical theory or provide definitions of such concepts here.

Morally ∞-cosmoi aim to describe the mathematical universe in which ∞-categories live, rather than prescribing to a particular model of ∞-categories. To those familiar with the model category literature you may notice that ∞-cosmoi can be realized as a kind of enriched category of fibrant objects.

Definition 18 (∞-Cosmos).

An **∞-cosmos** $\mathcal{K}$ is a category enriched over the full subcategory $\mathsf{QCat} \hookrightarrow \mathsf{sSet}$ of quasi-categories, together with a class of morphisms called *isofibrations* (denoted $\twoheadrightarrow$) such that

1. (**Completeness**) in the simplicially enriched sense $\mathcal{K}$ possesses
    1. a terminal object,
    2. small products,
    3. pullbacks of isofibrations,
    4. limits of countable towers of isofibrations,
    5. cotensors with simplicial sets
2. (**Isofibrations**) The class of isofibrations contains all isomorphisms and all maps $A \twoheadrightarrow 1$ (i.e. all objects are fibrant), while being closed under:
    1. composition,
    2. product,
    3. pullback,
    4. inverse limits of towers, and
    5. Leibniz cotensors with monomorphisms of simplicial sets (i.e. mimicking the notion of a closed model structure),

Isofibrations are also representably defined in the sense that if $f : A \twoheadrightarrow B$ is an isofibration and X is any object,

$$f_* : \mathrm{Fun}(X, A) \twoheadrightarrow \mathrm{Fun}(X, B)$$

is an isofibration of quasi-categories.

Moving forward we will write $\mathcal{K}$ for an abstract ∞-cosmos. If the reader wishes they can have in mind that $\mathcal{K}$ is the ∞-cosmos QCat of quasi-categories[^7, 1.2.10], where isofibrations are isofibrations of quasi-categories (i.e. inner fibrations that lift along boundary inclusions into the nerve of the free living isomorphism), or even the ∞-cosmos Kan of Kan complexes[^7, 1.2.12].

An important construction we will need when considering modules between ∞-categories is that of sliced ∞-cosmoi.

Definition 19 (Sliced ∞-Cosmos[^7, 1.2.22]).

For $B \in \mathcal{K}$, the **sliced ∞-cosmos** $\mathcal{K}_{/B}$ consists of

1. objects: isofibrations $p : E \twoheadrightarrow B$
2. functor spaces: from $p : E \twoheadrightarrow B$ to $q : F \twoheadrightarrow B$ defined by pullback

$$\mathrm{Fun}_B(p, q) := \{p\} \times_{\mathrm{Fun}(E,B)} \mathrm{Fun}(E, F)$$

3. isofibrations: commutative triangles with cone point B and all legs isofibrations
4. Terminal object $\mathrm{id}_B : B \twoheadrightarrow B$, and products given by pullback

$$\times_i^B E_i := B \times_{\prod_i B} \prod_I E_i$$

5. Pullbacks and limits of towers of isofibrations created by $\mathcal{K}_{/B} \to \mathcal{K}$
6. Simplicial cotensor of $p : E \twoheadrightarrow B$ with $U \in \mathsf{sSet}$ given by the pullback

$$U \pitchfork p := B \times_{B^U} E^U$$

Additionally, for those familiar with the essential role Kan complexes play in the quasi-categorical model of ∞-categories, it won't be surprising that an analogous class of objects in a general ∞-cosmos will play an essential role in formal ∞-category theory, particularly when discussing modules.

Definition 20 (Discrete ∞-Category).

An ∞-category $E \in \mathcal{K}$ is said to be **discrete** if for all $X \in \mathcal{K}$, $\mathrm{Fun}(X, E)$ is a Kan complex.

We can think of discrete ∞-categories as generalizations of ∞-groupoids as they can be equivalently characterized by the condition that all morphisms are isomorphisms[^7, 1.2.27], which is encoded by the statement that

$$E^{\mathbb{I}} \xrightarrow{\sim} E^{\mathbb{2}}$$

is a trivial isofibration.

3.2. Modules between ∞-Categories

Definition 21 (Module between ∞-Categories).

A **module** from A to B in $\mathcal{K}$ is a two-sided isofibration $E \xrightarrow{(q,p)} A \times B$ which is discrete as an object of $\mathcal{K}_{/A \times B}$ and such that $q : E \twoheadrightarrow A$ is cocartesian and $p : E \twoheadrightarrow B$ is cartesian.

This modules will take the place of our pro-arrows in a suitable generalization of pro-arrow equipments to the setting of the ∞-cosmos $\mathcal{K}$. In order to act as pro-arrows in an equipment we need that modules are suitably stable under base-change in the following sense.

Proposition 22 (Basechange of a Module [^7, 7.4.5]).

If $A \twoheadleftarrow E \twoheadrightarrow B$ is a module from A to B and $a : A' \to A$ and $b : B' \to B$ are a pair of functors, then the pullback defines a module $A' \twoheadleftarrow E(b, a) \twoheadrightarrow B'$

$$\begin{array}{ccc} E(b,a) & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \\ A' \times B' & \xrightarrow[a \times b]{} & A \times B \end{array}$$

Our prime example of modules will be those obtained from the comma category construction. Namely, the arrow ∞-category $A^{\mathbb{2}} \xrightarrow{(p_1, p_0)} A \times A$ is a module, implying that for any co-span $C \xrightarrow{g} A \xleftarrow{f} B$, the comma ∞-category $\mathrm{Hom}_A(f, g) = g \downarrow f$:

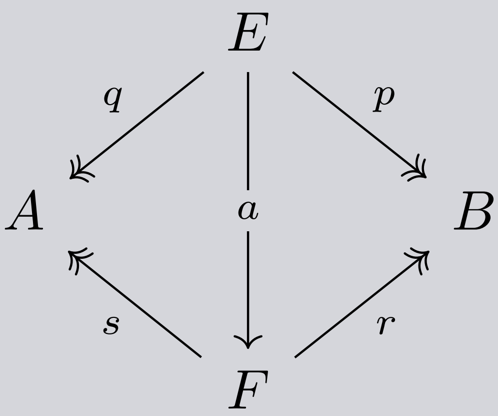
$$\begin{array}{ccc} g \downarrow f & \longrightarrow & A^{\mathbf{2}} \\ \big\downarrow & & \big\downarrow (p_1, p_0) \\ C \times B & \xrightarrow[f \times g]{} & A \end{array}$$

where $g \downarrow f$ is used to indicate that it is a module from C to B.

Now that we have our modules, or pro-arrows, we want to define a double category which will be our pro-arrow equipment we can do formal ∞-category theory internal to. However, as we will see shortly this isn't strictly possible for a general ∞-cosmos due to the fact that the natural composition of modules does not preserve the property of being discrete. In order to correct this we would need to require that our ∞-cosmoi have suitable weak colimits. Instead of leaving their axiomatic construction at this point in order to obtain a pro-arrow equipment of modules, Riehl and Verity instead construct a weaker structure, known as a *virtual equipment*, as a full subcategory of a pro-arrow equipment of two-sided isofibrations.

Definition 23 (Fibered Maps).

A **fibered map of two-sided isofibrations** from $A \xleftarrow{q} E \xrightarrow{p} B$ to $A \xleftarrow{s} F \xrightarrow{r} B$ is a fibered isomorphism class of strictly commuting diagrams



where two such diagrams with a and a' are considered equivalent if there exists a natural isomorphism $\gamma : a \cong a'$ such that $s\gamma = \mathrm{id}_q$ and $r\gamma = \mathrm{id}_p$.

We can view the 1-category of isofibrations from A to B in an ∞-cosmos $\mathcal{K}$ as a quotient of the homotopy 2-category $h(\mathcal{K}_{/A\times B})$. This 1-category exactly captures the notion of equivalence in $\mathcal{K}_{/A\times B}$ in the sense that a pair of two-sided isofibrations are equivalent in $\mathcal{K}_{/A\times B}$ if and only if they are isomorphic in this one-category.

We can now define the double category which are virtual equipment will live inside

Definition 24 (Double Category of Two-Sided Isofibrations).

The homotopy 2-category of an ∞-cosmos $\mathcal{K}$ supports a (non-unital) **double category of two-sided isofibrations** whose:

1. Objects are ∞-categories
2. vertical arrows are functors
3. horizontal arrows $E : A \twoheadrightarrow B$ are two-sided isofibrations
4. and whose squares with boundary $f : A' \to A$ and $g : B' \to B$ are maps of two-sided isofibrations over $f \times g$.

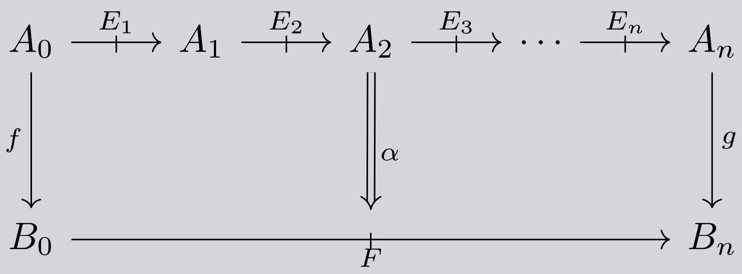
This double category generally does not have a horizontal unit since the identity span is not generally a two-sided isofibration. However, when we restrict to the virtual double category of modules we will realize a unit from the arrow category. First let's recall the notion of a virtual double category, which can be thought of as an analogue of multicategories in the double categorical world.

Definition 25 (Virtual Double Category).

A **virtual double category** consists of

1. A category $\mathbb{D}_0$ of objects and vertical arrows
2. For any pair of objects A and B in $\mathbb{D}_0$ a collection of *horizontal arrows* $A \twoheadrightarrow B$

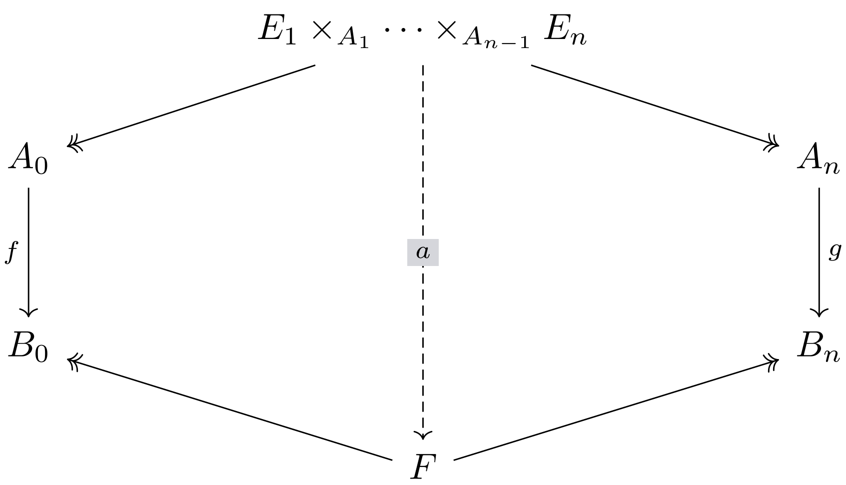
3. Cells with boundary



and composition given by pasting.

4. an **identity cell** for vertical composition for every horizontal arrow $E : A \nrightarrow B$.

From our double category of two-sided isofibrations we can obtain a virtual double category with the same objects, vertical arrows, and 1-ary squares, as well as n-ary cell of maps of two-sided isofibrations over $f \times g : A_0 \times A_n \to B_0 \times B_n$ given by



whose vertical source is the $(n-1)$-fold span composite of the sequence of spans $E_1, \ldots, E_n$ [7, 8.1.11]. This construction gives us a unit, in the sense of virtual double categories, which is a 0-ary square with edges $\mathrm{id}_A : A \to A$ and horizontal two-sided isofibration $A^{\mathbb{I}} \xrightarrow{(q_1,q_0)} A \times A$.

The virtual double category of modules can now be given as follows [7, 8.1.14]:

Definition 26 (Virtual Double Category of two-sided isofibrations).

The **virtual double category of modules** $\mathbb{M}od(\mathcal{K})$ is the full subcategory of the virtual double category of isofibrations whose

1. objects are ∞-categories
2. vertical arrows are functors
3. horizontal arrows $E : A \nrightarrow B$ are modules from A to B

4. n-ary cells are fibered isomorphism classes of maps of two-sided isofibrations over $f \times g$, as above.

Note the reason we needed to pass to virtual double categories is that $E_1 \times_{A_1} \cdots \times_{A_{n-1}} E_n$, although being a two-sided isofibration, need not be a module (as it may fail to be discrete in $\mathcal{K}/(A_0 \times A_n)$).

In order to define a notion of Kan extension in $\mathbb{M}od(\mathcal{K})$ it remains to show that this virtual double category has the structure of an virtual equipment.

Definition 27 (Virtual Equipment).

A **virtual equipment** is a virtual double category such that

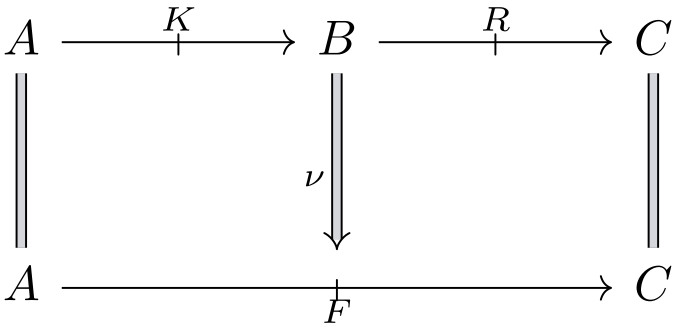
1. for any horizontal arrow $E : A \twoheadrightarrow B$ and any pair of vertical arrows $a : A' \to A$ and $b : B' \to B$, there exists a horizontal arrow $E(b, a) : A' \twoheadrightarrow B'$ and a unary cartesian cell ρ satisfying the natural universal property
2. every object A admits a **unit** horizontal arrow $\mathsf{Hom}_A : A \twoheadrightarrow A$ equipped with a nullary cocartesian cell ι satisfying the natural universal property (i.e. all n-ary cells with object A in their domain factor uniquely through ι and identities on the other components of the domain).

In $\mathbb{M}od(\mathcal{K})$, Hom_A is exactly the arrow category $A^{\mathbb{2}}$ as a module, while for $g : C \to A$ and $f : B \to A$, we write $\mathsf{Hom}_A(f, g)$ for the base change given in Proposition 22 (Basechange of a Module [^7, 7.4.5]). The nullary cell ι is determined by a map $A \to \mathsf{Hom}_A = A^{\mathbb{2}}$ given by picking out the identity $\mathrm{id}_A : A \to A$. On the other hand, the unary cartesian cell in $\mathbb{M}od(\mathcal{K})$ is given as in Proposition 22 (Basechange of a Module [^7, 7.4.5]). The modules $\mathsf{Hom}_B(f, B)$ and $\mathsf{Hom}_B(B, f)$ define conjoints and companions for $f : A \to B$, respectively.

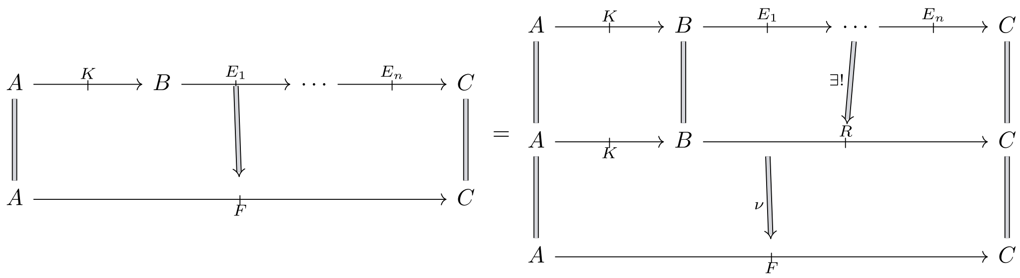
3.3. Extensions Via Modules

We can now begin formalizing Kan extensions in $\mathbb{M}od(\mathcal{K})$. This can be done using the naive theory of right extensions in any bicategory, where the virtual bicategory we will be considering is $\mathsf{Ver}(\mathbb{M}od(\mathcal{K}))$. Since virtual bicategories take a slightly different form let's make this universal property explicit.

A **right extension** of a module $F : A \twoheadrightarrow C$ along a module $K : A \twoheadrightarrow B$ consists of a module $R : B \twoheadrightarrow C$ together with a binary cell



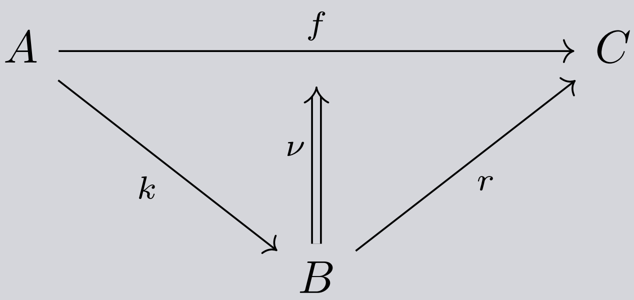
which is universal in the sense that for $(n+1)$-ary cell of the form below-left, we have a unique factorization into a composite below right:



We can now finally define what it means for to have a pointwise right Kan extension in $\mathcal{K}$ [7, 9.3.3]:

Definition 28 (Pointwise Right Extension).

A diagram of the form



in $h\mathcal{K}$ is a pointwise right Kan extension if and only if $\nu_* : \mathsf{Hom}_B(B, k) \times. \mathsf{Hom}_C(C, r) \Rightarrow \mathsf{Hom}_C(C, f)$ defines a right extension in $\mathsf{Ver}(\mathbb{M}od(\mathcal{K}))$.

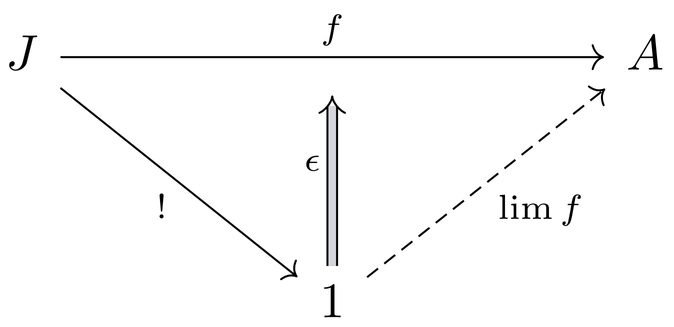
In order to get left Kan extensions we use the co-dual notion of right lifts in $\mathsf{Ver}(\mathbb{M}od(\mathcal{K}))$. The reason that these are pointwise is that the extensions defined in this way are stable under pasting with exact squares (or equivalently comma squares), which corresponds classically to the condition that the Kan extensions are stable under post-composition with representables.

With the notion of pointwise right Kan extensions we can immediately start doing formal ∞-category theory in $\mathbb{M}od(\mathcal{K})$. This will be expanded on in a future note for a future talk, but for now let's cite some interesting constructions and results we obtain with this theory.

Facts:

1. Pointwise right extensions along right adjoints exist and are given by restriction along the left adjoint
2. Taking a right extension along a fully faithful functor (as defined in a pro-arrow equipment with horizontal identities Hom_A) has as structure transformation an isomorphism
3. A right adjoint is fully faithful if and only if its counit is an isomorphism

Limits in $\mathcal{K}$ agree with the following definition: A **limit** of a diagram of ∞-categories $f : J \to A$ is a pointwise right extension as below right



which agree with the classical definition in cartesian closed ∞-cosmoi. More generally, given a module $W : A \nrightarrow B$ and a functor $f : A \to C$, we can say that a functor $\lim^W f : B \to C$ defines a **W-weighted limit** of f if it covariantly represents the right extension of $\mathrm{Hom}_C(c, f)$ along W. (i.e. $\lambda : W \times_{.} \mathrm{Hom}_C(C, \lim^w f) \Rightarrow \mathrm{Hom}_C(C, f)$ is a right Kan extension).

3.3.1. References

1 Ruit, J. (2024, August 26). Formal category theory in ∞-equipments I. arXiv. https://doi.org/10.48550/arXiv.2308.03583
2 Riehl, E., & Verity, D. (2015, May 6). The 2-category theory of quasi-categories. arXiv. https://doi.org/10.48550/arXiv.1306.5144
3 Shulman, M. A. (2009, January 9). Framed bicategories and monoidal fibrations. arXiv. https://doi.org/10.48550/arXiv.0706.1286
4 Riehl, E., & Verity, D. (2017). Kan extensions and the calculus of modules for ∞–categories. Algebraic & Geometric Topology, 17(1), 189–271. https://doi.org/10.2140/agt.2017.17.189
5 Riehl, E., & Verity, D. (2016, June 13). Fibrations and Yoneda's lemma in an ∞-cosmos. arXiv. https://doi.org/10.48550/arXiv.1506.05500
6 Verity, D. (1992). Enriched categories, internal categories and change of base. https://research.ebsco.com/linkprocessor/plink?id=3d6aa350-468b-3617-91c2-8c311657660b

7 Riehl, E., & Verity, D. (2022). Elements of ∞-Category Theory. Cambridge: Cambridge University Press. https://doi.org/10.1017/9781108936880